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AN INTRODUCTION TO HARMONIC ANALYSIS OF MAGNETS

by

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Two Dimensional Magnetic Fields

1.1 Representation By Complex Numbers

In what follows, only two dimensional magnetic fields and their harmonic analysis will be considered. In most applications to accelerator magnets, this approximation will suffice. In two dimensions, the static Maxwell's equations state

$$\mathrm{div}\,\vec{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

$$\mathrm{curl}\,\vec{B} = 0 \Rightarrow \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0$$

However, if W=x+iy and a function f(W) is analytic in W, the Cauchy-Reimann equations imply

$$\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} = 0 \;; \quad \text{and} \quad \frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y} = 0$$

where $f=f_x+if_y$.

This implies that $B \equiv B_x+iB_y$ cannot be analytic in W . However, it also implies that B is analytic in W^* , the complex conjugate of W .

A general two dimensional magnetic field can be expanded in powers of W^* .

$$B(W^*) = \sum_{k=1}^{\infty} i\, G_k (W^*)^{k-1} \quad ; \quad i = \sqrt{-1}$$

The coefficients G_k are known as the harmonics of pole 2k. e.g G_1 is the dipole harmonic. The dipole harmonic field is constant with respect to W^* . G_2 is the quadrupole harmonic. The field due to the quadrupole varies linearly with respect to W^* .

The G_k are in general complex since B is a general function of W^* . The real part of G_k is known as the normal harmonic and the imaginary part of G_k is known as the skew harmonic. For instance, if B is a pure quadrupole field, all G_k are zero except G_2 , so B may be expressed.

$$B = i\, G_2\, W^*$$

$G_2 = G_N + i\, G_S$ where G_N and G_S are the normal and skew quadrupole harmonics respectively. Then if $W = r e^{i\phi}$

$$B \equiv B_x + i B_y = i\,(G_N + i G_S)\, r\, e^{-i\phi}$$

yielding

$$B_x = r\,(G_N \sin\phi - G_S \cos\phi)$$

$$B_y = r\,(G_N \cos\phi + G_S \sin\phi)$$

1.2 Conventions

The magnets are measured from the downstream end. i.e. we look at the magnet in such a way that the direction of the positive beam leaves the magnet and strikes the eye. In this co-ordinate system, the y axis is drawn so that +y is vertical up and +x is horizontal to the right. Positive theta is a counter-clockwise rotation.

The definition of normal and skew harmonics used in this paper are related to those employed by the Magnet Test Facility (MTF) in a simple way. MTF defines

$$B_y + i B_x = \sum_{k=1}^{\infty} \left(g_n^k + i g_s^k \right) (x + iy)^{k-1}$$

Comparing the two definitions, it is easy to show

$$G_k \equiv \left(G_N^k + i G_s^k \right) = \left(g_n^k - i g_s^k \right)$$

$$\text{i e} \quad G_N^k = g_n^k \quad \text{and} \quad G_s^k = - g_s^k$$

i.e the normal harmonics are defined the same way. There is a sign change in the skew harmonic. This is carried over to the definition of harmonic moments. (see below)

1.3 Definition of Harmonic Moments

The harmonics defined in Section I are not solely dependent on the properties of the magnet in question, but also on the amount of current flowing through it. In order to arrive at quantities solely dependent on the magnet geometry, harmonic moments are defined.

For a magnet of primary pole 2p (a dipole has p=1, quadrupole has p=2 and so on), the harmonic moment M_K is defined as

$$M_K \equiv \frac{G_K}{|G_p|} r_0^{k-p}$$

where r_o is a reference radius usually taken to be 1".

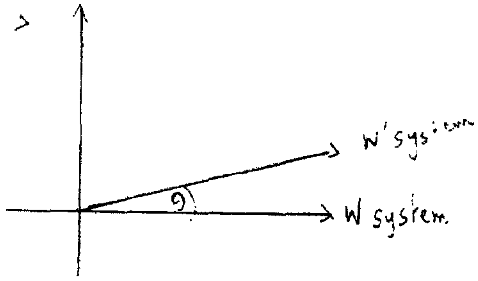
M_K is in general complex; its real part is the normal moment and the imaginary part is the skew moment. M_K is independent of the current.

M_K is in fact the magnetic field due to the k^{th} harmonic at a point r_o on the real axis normalised to the maximum field due to the primary pole on a circle of radius r_o .

1.4 Transformation Properties of Harmonics

Consider a transformation of co-ordinates such that

$W' = W e^{-i\theta}$.



This corresponds to a rotation by θ in a counter-clockwise direction. During this rotation, a vector such as W is trasformed to W' as given by the equation $W' = W e^{-i\theta}$. Consider a magnetic field $B_K = i\, G_K (W^*)^{K-1}$. Then B_K being a vector has to transform similarly. i.e.

$$
\begin{aligned}
B'_K &= B_K\, e^{-i\theta} \\
&= i\, G_K (W^*)^{K-1}\, e^{-i\theta} \\
&= i\, G_K \left(W'^{*} e^{-i\theta}\right)^{K-1} e^{-i\theta} \\
&= i\, G_K \left(W'^{*}\right)^{K-1} e^{-iK\theta} \\
&\equiv i\, G'_K \left(W^{*\prime}\right)^{K-1}
\end{aligned}
$$

$$\therefore \quad G'_K = G_K\, e^{-iK\theta}$$

i.e. under a rotation, the new harmonic G'_K is given by $G_K\, e^{-iK\theta}$.

If G_K is real, it is a normal harmonic. One can ask, what angle does one have to rotate a magnet till a normal magnet becomes a skew magnet ? For this to happen,

$$\frac{G'_K}{G_K} = e^{-i\pi/2} = e^{-iK\theta} = -i$$

$$\therefore \quad \theta = \pi/2K$$

Thus a dipole has to be rotated through 90 degrees, a quadrupole through 45 degrees, a sextupole through 30 degrees and so on for a normal magnet to become a skew magnet.

1.5 Definition of Magnetic centers

The polynomial expansion

$$B = \sum_{k=1}^{N+1} i\, G_k \left(W^*\right)^{k-1}$$

can be used to approximate a magnetic field in a region of interest within which the expansion is deemed valid. The magnetic center is defined to be that point at which B =0. The above expansion of B is a polynomial of degree N. Clearly there will be N values of W for which B = 0. In general, not all these values of W will lie within the region of interest. For instance, in the çase of the dipole magnet, none of the N values of W will lie within the region of interest. For the quadrupole, sextupole etc., there will be one and only one such value of W at which the field is zero within the magnet volume. That point is defined as the magnetic center.

1.6 Effect of being off-center.

Consider a primary pole 2p

$$B_p = i\, G_p \left(W^*\right)^{p-1}$$

Clearly the center of the above field is at W = 0. If one now transforms the measuring system such that W′ = W + $\underline{a}$, then in this system the center is at W′ = $\underline{a}$. Then,

$$\begin{aligned} B_p' &= i\, G_p \left(W^{*\prime} - \underline{a}^*\right)^{p-1} \\ &= i\, G_p \left(W^{*\prime}\right)^{p-1} - i\,(p-1)\, G_p\, \underline{a}^* \left(W^{*\prime}\right)^{p-2} \end{aligned}$$

to first order in $\underline{a}$.

Then in this system, there will be a harmonic G_{p-1} given by

$$G_{p-1} = -(p-1)\, G_p\, \underline{a}^*$$

$$\therefore\ M_{p-1} \equiv \frac{G_{p-1}}{|G_p|}\, r_0 = -(p-1)\, \frac{G_p}{|G_p|}\, \underline{a}^*\, r_0$$

where M_p is the p-1th harmonic moment.

$$\therefore\ \underline{a}^* = -\frac{M_{p-1}}{(p-1)\, r_0}\ \frac{|G_p|}{G_p}$$

Thus, by measuring the p-1th moment, the center $\underline{a}$ can be determined for small displacements $\underline{a}$. If all $\bar{N}$ harmonics are determined, the center may alternately be found by solving the polynomial equation B = 0.

The following note is an attempt to clear up some misunderstandings about the effects of being off center. This is perhaps best done by giving a specific example. For a quadrupole magnet, being off center will induce a dipole harmonic. Also, if the magnet symmetry is slightly distorted, a dipole term will be present. People then worry; is the dipole term being measured produced by being off the magnetic center or is it produced by symmetry violations in the magnet fabrication?. The question is basically meaningless as can be illustrated by the following. Consider a perfect quadrupole whose magnetic center is also its geometric center. Now introduce symmetry breaking distortions in the magnet that produce a dipole term. Thus the new field can be written:

$$B = i\, G_1\, W^* + i\, G_0$$

where G_0 is the new dipole term.

But the net effect of the dipole term thus introduced is to shift the magnetic center away from the geometric center of the magnet. For B can be re-written: $B = i\, G_1 \left(W^* + \frac{G_0}{G_1}\right)$

or $B = i\, G_1\, W'^*$

where $W' = W + \left(\frac{G_0}{G_1}\right)^*$

i.e. the magnetic center is now at $W' = 0$ or $W = -\left(\frac{G_0}{G_1}\right)^*$.

So if we now remain at the geometric center, we will measure the dipole term. But is it due to the symmetry breaking effects or due to the fact that we are off the magnetic center? The answer is that the symmetry breaking effects can be absorbed into a term describing a displacement from the magnetic center!

1.7 Transformation of Harmonics under a rotation and translation

Consider a transformation of co-ordinates such that

$$W' = \alpha W + \beta \qquad |\alpha| = 1 \quad \text{i.e} \quad \alpha = e^{-i\theta}$$

W' is the co-ordinate measured in a system rotated counter clockwise by angle θ and translated by $-\beta$ from the W co-ordinate system.

$$B = \sum_{k=1}^{\infty} i\, G_k (W^*)^{k-1}$$

$$B' = \sum_{k=1}^{\infty} i\, G_k' (W'^*)^{k-1} = \alpha B$$

$$\therefore \sum G_k' (W'^*)^{k-1} = \alpha \sum G_k (W^*)^{k-1} = \alpha \sum G_k \left(\frac{W'^* - \beta^*}{\alpha^*}\right)^{k-1}$$

Equating coefficients of $(W'^*)^k$ we get G_k' in terms of G_k.
Let

$$p = -\beta^* \;;\quad q = (W^*)' \;;\quad r = \alpha^*$$

Then $$\sum_{k=1}^{\infty} G_k' \frac{q^{k-1}}{\alpha} = \sum G_k \frac{(p+q)^{k-1}}{r^{k-1}}$$

expanding the RHS, we get

$$\frac{G_1'}{\alpha} = G_1 + G_2 \frac{p}{r} + G_3 \frac{p^2}{r^2} + G_4 \frac{p^3}{r^3}$$

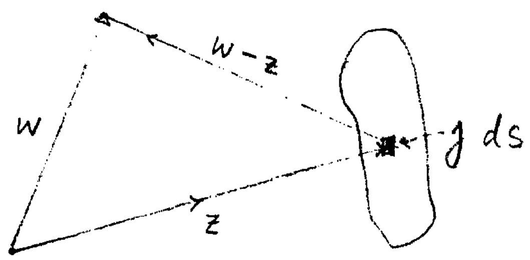
$$\frac{G_2'}{\alpha} = \frac{1}{r}\left(G_2 + 2G_3 \frac{p}{r} + 3G_4 \frac{p^2}{r^2} \cdots \right)$$

in general

$$\boxed{G_i' = \alpha \sum_{k=i}^{\infty} \frac{(k-1)!}{(i-1)!\,(k-i)!} \frac{p^{k-i}}{r^{k-1}} G_k}$$

Note the sum on RHS runs from $i \rightarrow \infty$

1.8 Calculation of Magnetic Fields from Current Distributions.

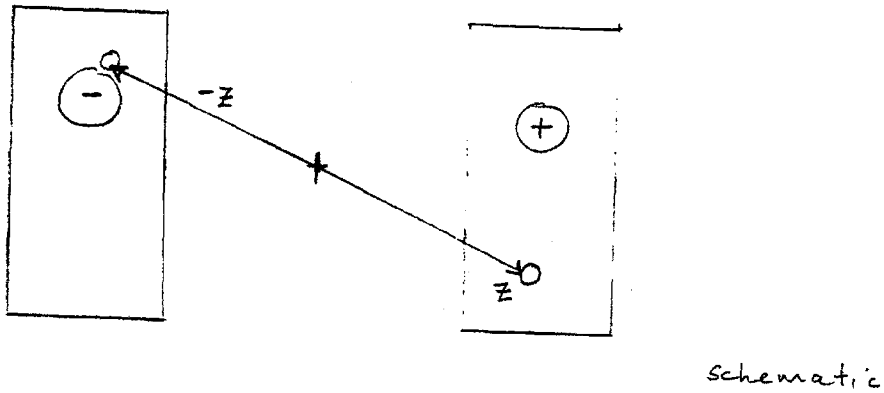


Consider a current element jds at a position given by Z. What is the magnetic field dB due to this current element at a position W ? This can easily be shown to be

$$dB = -i \frac{\mu_0 j\,ds}{2\pi} \frac{1}{(W^* - Z^*)}$$

The field due to an arbitrary current distribution j(Z) can be calculated by integrating the above equation.

a) Dipole Magnets



Consider now a current distribution where for every j(Z) there is a j(-z) such that

$$j(z) = -j(-z)$$

This is the case for a typical dipole magnet. Then the field due to the dipole magnet can be written

$$\int dB = -\frac{i\mu_0}{2\pi} \int_{Re\, z>0} \left(\frac{1}{w^* - z^*} - \frac{1}{w^* + z^*} \right) j(z)\, ds$$

$$= -\frac{i\mu_0}{2\pi} \int \frac{2z^* \; j(z)\, ds}{(w^*)^2 - (z^*)^2}$$

$$= \frac{i\mu_0}{2\pi} \int \frac{2z^* \, j(z)\, ds}{(z^*)^2 \left(1 - \frac{(w^*)^2}{(z^*)^2} \right)}$$

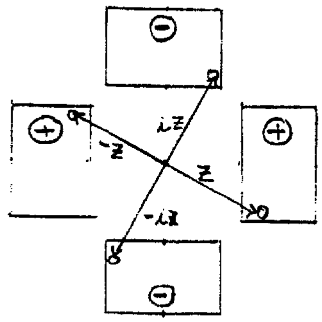
$$= \frac{i\mu_0}{2\pi} \int \frac{2 \, j(z)\, ds}{z^*} \left(1 + \frac{(w^*)^2}{(z^*)^2} + \frac{(w^*)^4}{(z^*)^4} + \frac{w^{*6}}{(z^*)^6} \cdots \right.$$

if $|w| < |z|$

$$= A + B(w^*)^2 + C(w^*)^4 + D(w^*)^6$$

i.e the field due to a dipole magnet with perfect symmetry does not contain odd powered harmonics. i.e no quadrupole, octupole 12 pole etc. Only 6 pole, 10 pole 14 pole etc may occur.

b) Quadrupole Magnets.



Quadrupole magnets have current distributions such that

$$j(z) = -j(iz) = j(-z) = -j(-iz)$$

$$\therefore \int dB = -\frac{i\mu_0}{2\pi} \int \left[\frac{1}{w^* - z^*} - \frac{1}{w^* + i z^*} + \frac{1}{w^* + z^*} - \frac{1}{w^* - i z^*} \right] j(z)\, d$$

$$= \sum_{k=0}^{\infty} a_k \left(w^*\right)^{4k+1}$$

So quadrupoles with perfect symmetry can only have the 12 pole, 20 pole, 28 pole etc harmonics. Similarly the harmonics occuring in higher poles can be worked out. In general, a magnet with primary pole 2p with perfect symmetry can only have harmonics $2p(2k-1)$; $k = 1, 2, 3 \ldots \infty$

c) Arbitrary Shaped Areas with constant current density.

Very often, the current density is constant within a given area of arbitrary shape. For this special case, the surface integral in two dimensions can be reduced to an integral around the contour of the area in question. For this purpose, we employ the two dimensional version of Stokes' theorem which can be stated

$$\oint A\, dz^* = -2i \iint \frac{dA}{dz}\, ds$$

where A is an analytic function of Z.

We wish to calculate,

$$B = -\frac{i\mu_0}{2\pi} \iint \frac{J\,dS}{W^* - Z^*}$$

Taking the complex conjugate of Stokes' equation

$$\oint A^* dZ = 2i \iint \left\{\frac{dA}{dZ}\right\}^* dS$$

$$2i\left(\frac{dA}{dZ}\right)^* = -\frac{i\mu_0}{2\pi}\frac{J}{W^* - Z^*}$$

$$\therefore \frac{dA}{dZ} = -\frac{\mu_0}{4\pi}\frac{J}{W - Z} = \frac{\mu_0 J}{4\pi}\frac{1}{Z\left(1 - \frac{W}{Z}\right)}$$

$$= \frac{\mu_0 J}{4\pi Z}\left[1 + \frac{W}{Z} + \frac{W^2}{Z^2} \cdots\right]$$

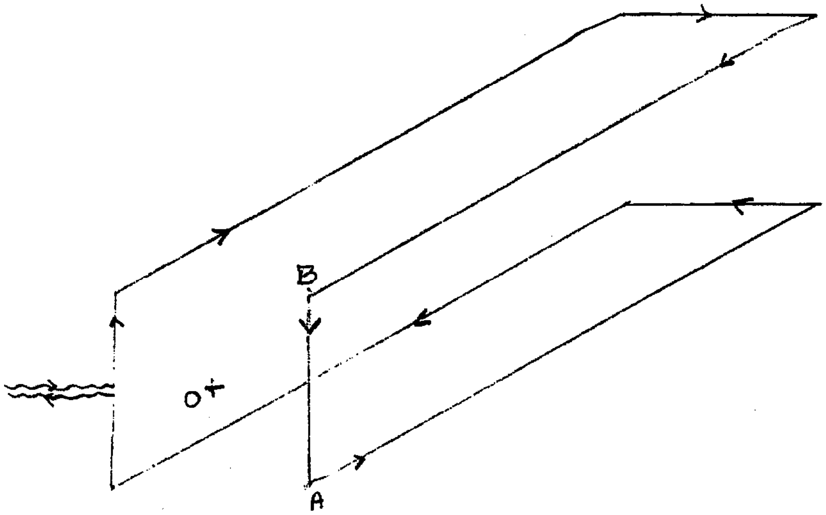
$$\therefore A = \frac{\mu_0 J}{4\pi}\left\{\ln Z - \sum_{k=1}^{\infty}\frac{W^k}{kZ^k}\right\}$$

$$\therefore B = \oint A^* dZ = \frac{\mu_0 J}{4\pi}\left\{\oint \ln Z^* dZ - \sum_{k=1}^{\infty}\oint\frac{dZ\,(W^*)^k}{k(Z^*)^k}\right.$$

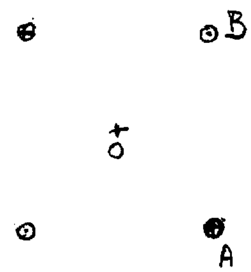
The above equation is the harmonic expansion of the field due to an arbitrary shaped conductor. In practice, while working out the dipole term on a computor, one has to be careful that the conductor does not cross the negative real axis, since the cut in ln(Z) on the negative axis can lead to spurious results. In cases where such a crossing is inevitable, it may be advisable to translate the conductor to the positive real axis, work out the dipole contribution and translate back.

1.9 Morgan Coil Pickup Equations.

A Morgan Coil is a device of a particular harmonic symmetry such that it is sensitive to fields having that particular harmonic and its odd multiples only. To illustrate, a quadrupole Morgan Coil will look as follows:

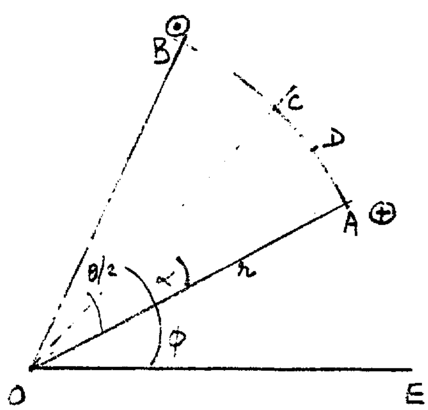


Side view



Edge view

We will work out the flux suspended by a Morgan Coil of symmetry 2n in a field of symmetry 2k. e.g for a quadrupole Morgan Coil in a sextupole field n=2, k=3.



The radius of the Morgan Coil =r. The angle suspended at the center by the two consecutive legs $= \theta = \frac{\pi}{n}$. The bisector OC between the two legs is at an angle ϕ about the zero reference direction OE.

We wish to calculate the field at D and integrate the radial component of the field along AB to get the flux subtended by one sector of the Morgan Coil . The field is given by

$$B_K = i G_K (W^*)^{k-1} \quad , \quad W = r e^{i\delta} \quad \text{where } \delta = \phi - \frac{\theta}{2} + \alpha$$

The component of B_K along the radius $e^{i\delta}$ is given by

$$B_r = \text{Real part of } [B_K e^{-i\delta}]$$
$$= Re\left[i G_K r^{k-1} e^{-ik\delta}\right]$$

Therefore the flux through the leg = $\displaystyle \Phi_K = \int_{\delta=\phi-\frac{\theta}{2}}^{\phi+\frac{\theta}{2}} r L B_r \, d\delta$

where L = length of Morgan Coil .

$$\Phi_K = Re\int r L i G_K r^{k-1} e^{-ik\delta} d\delta$$
$$= Re \int i G_K r^K L e^{-ik\delta} d\delta$$
$$= Re \; i G_K r^K L \left[\frac{e^{-ik\delta}}{-ik}\right]_{\delta=\phi-\frac{\theta}{2}}^{\phi+\theta/2}$$
$$= Re\left(2i G_K r^K L \frac{\left(e^{ik\frac{\theta}{2}} - e^{-ik\frac{\theta}{2}}\right)}{2ik} e^{-ik\phi}\right); \quad \theta = \pi/n$$
$$= Re\left(\frac{2i G_K r^K L}{k} e^{-ik\phi} \sin\frac{k\pi}{2n}\right)$$

To get the total flux through the coil, we have to sum over every other sector of the coil. To see this , consider the total area subtended by the coil and integrate the field over it taking care to remain on the same side of the subtended area.

Then the sectors which are to be summed over are at angles ϕ_λ given by

$$\phi_\lambda = \phi_1 + (\lambda - 1)\frac{2\pi}{n} \qquad \lambda = 1, 2 \ldots n$$

Therefore total flux

$$\Phi_k^n = \frac{2r^k L}{k} \sin\frac{k\pi}{2n} \, Re\left\{ i G_k S \right\}$$

$$\text{where } S = e^{-ik\phi_1} \sum_{\lambda=1}^{n} e^{-ik\frac{(\lambda-1)2\pi}{n}}$$

Consider the case when n = k. i.e. Morgan Coil has the same symmetry as the field.

In this case,

$$S = n\, e^{-in\phi_1}$$

$$\Phi_n^n = 2r^n L \; Re\left\{ i G_k e^{-in\phi_1} \right\}$$

In the above equation, ϕ_1 is the angle of the bisector of the first sector. The angle ϕ of the first leg = $\phi_1 - \frac{\pi}{2n}$.

$$\therefore \Phi_n^n = 2r^n L \; Re\left\{ G_K e^{-in\phi} \right\}$$

$$= 2r^n L \left\{ G_n \cos n\phi + G_s \sin n\phi \right\}$$

where $G_K = G_n + iG_s$, G_n and G_s are the normal and skew components of the field respectively.

The case $k \neq n$.

When the Morgan Coil symmetry is not the same as the field symmetry, the flux suspended is in most cases zero except for those cases where k is an odd multiple of n. To see this, examine the behavior of the sum

$$S = e^{-ik\phi_1} \sum_{\lambda=1}^{n} e^{-ik(\lambda-1)\frac{2\pi}{n}}$$

and of $\sin\left(\frac{k\pi}{2n}\right)$

When k is an odd multiple of n, one can show that,

$$\Phi_k^n = \frac{2r^k L n}{k} \left[G_n^k \cos k\phi + G_s^k \sin k\phi \right]$$

with $G_k \equiv G_n^k + i G_s^k$

So a dipole Morgan Coil is sensitive to a sextupole field but not to a quadrupole field.

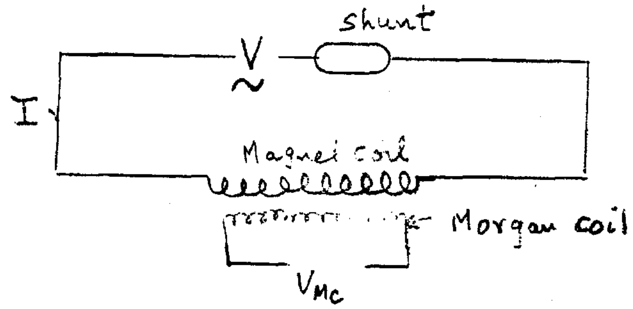
1.10 Transfer constants

Using a Morgan Coil with various windings, one can perform harmonic analysis of magnets. Also, it is possible to derive $\int Bdl$ for the magnets at a fixed radius r_o .

The flux subtended by a field of symmetry 2n on a Morgan Coil of symmetry 2n is given by

$$\Phi_n^n = 2r^n L \left\{ G_n \cos n\phi \right\}$$

assuming no skew term is present, for the sake of analysis.



To measure the transfer constant (another term for $\int Bdl$ at r_0), the above circuit is used. The voltage across the shunt is measured to obtain the current in the magnet coil. The voltage across the Morgan Coil is measured to obtain the field produced by the magnet.

The voltage induced in the Morgan Coil $= V_{MC} = -\dfrac{d\Phi_n^n}{dt}$

$$\therefore V_{MC} = 2r^n L \dot{G}_n \cos n\phi$$

The field $B_n = i\, G_n (w^*)^{n-1}$

The harmonic G_n is proportional to the current I.

$$G_n = G_n^0 \left(\frac{I}{I_0}\right)$$

where I_0 is the standard current usually taken to be 50 Amps.

$$\therefore V_{MC} = 2r^n L \frac{G_n^0}{I_0} \left(\frac{dI}{dt}\right) \cos n\phi$$

$I(t) = I_{max} \cos \omega t$; ω = AC frequency

$$\therefore V_{MC} = 2r^n L \frac{G_n^0}{I_0} \omega I_{max} \sin \omega t \cos n\phi$$

$$\therefore V_{MC}^{Max} = 2r^n L \frac{G_N^0}{I_0} \omega I_{max}$$

The current I_{Max} can be found by measuring the shunt voltage.

$$I_{max} = \frac{V^{max}_{shunt}}{R_{shunt}}$$

$$\therefore \; G^{o}_{n} = \frac{V^{Max}_{Mc} \; I_{o}}{2 r^{n} L \omega} \; \frac{R_{shunt}}{V^{Max}_{shunt}}$$

$$\therefore \; B^{o}_{n} = G^{o}_{n} (r^{o})^{n-1} \equiv \text{maximum field on a circle of radius } r_{o}.$$

$$\therefore \int B \, dl \text{ at } r_{o} = B^{o}_{n} \, L \quad \text{(assuming Morgan coil is longer than Magnet)}$$

$$= \quad Q \quad = \frac{V^{Max}_{Mc} \; I_{o} \; R_{shunt}}{2 r \omega \, V^{Max}_{shunt}} \left(\frac{r_{o}}{r}\right)^{n-1}$$

The above expression in RMKS will yield Q in Tesla metres if ω is in Hz, R_{shunt} is in Ohms, and r is in metres.

to convert to kG" remember that

10kG = 1 Tesla

1" = 2.54×10^{-2} metres

1kG" = 2.54×10^{-3} Tesla metres

Q should be divided by 2.54×10^{-3} to get the answer in kG in.

REFERENCES

For additional reading, please consult:

1) "Stationary coil for measuring the harmonics in Pulsed Transport magnets" G.H.Morgan, Proceedings of the 1962 Magnet Conference at Brookhaven National Laboratory. page 787

2) Some Analytic Methods for winding configuration of ironless beam transport magnets and lenses. A.Asner and C.Iselin, Proceedings of the 2nd International conference on Magnet Technology, Oxford(1967) page 32.